

PLDA with Two Sources of Inter-session Variability

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1 The Model

1.1 PLDA

We take a linear-Gaussian generative model \mathcal{M} . We suppose that we have i-vectors of the same conversations recorded simultaneously by different channels or different noisy conditions. Then, an i-vector ϕ_{ijk} of speaker i , session j recorded in a channel l can be written as:

$$\phi_{ijl} = \mu + \mathbf{V}\mathbf{y}_i + \mathbf{U}\mathbf{x}_{ij} + \epsilon_{ijl} \quad (1)$$

where μ is speaker independent term, \mathbf{V} is the eigenvoices matrix, \mathbf{y}_i is the speaker factor vector, \mathbf{U} is an the eigenchannels matrix, \mathbf{x}_{ij} and ϵ_{ijl} is a channel offset. The term \mathbf{x}_{ij} must be the same for all the recordings of the same conversation. The term ϵ_{ijl} accounts for the channel variability.

We assume the following priors for the variables:

$$\mathbf{y} \sim \mathcal{N}(\mathbf{y}|\mathbf{0}, \mathbf{I}) \quad (2)$$

$$\mathbf{x} \sim \mathcal{N}(\mathbf{x}|\mathbf{0}, \mathbf{I}) \quad (3)$$

$$\epsilon \sim \mathcal{N}(\epsilon|\mathbf{0}, \mathbf{W}^{-1}) \quad (4)$$

where \mathcal{N} denotes a Gaussian distribution; and \mathbf{D} is a full rank precision matrix. ϕ is an observable variable and \mathbf{y} and \mathbf{x} are hidden variables.

1.2 Notation

We are going to introduce some notation:

- Let Φ_d be the development i-vectors dataset.
- Let $\Phi_t = \{l, r\}$ be the test i-vectors.
- Let Φ be any of the previous datasets.
- Let θ_d be the labelling of the development dataset. It partitions the N_d i-vectors into M_d speakers. Each speaker has H_i sessions and each session can be recorded by L_{ij} different channels.
- Let θ_t be the labelling of the test set, so that $\theta_t \in \{\mathcal{T}, \mathcal{N}\}$, where \mathcal{T} is the hypothesis that l and r belong to the same speaker and \mathcal{N} is the hypothesis that they belong to different speakers.
- Let θ be any of the previous labellings.
- Let Φ_i be the i-vectors belonging to the speaker i .

- Let \mathbf{Y}_d be the speaker identity variables of the development set. We will have as many identity variables as speakers.
- Let \mathbf{Y}_t be the speaker identity variables of the test set.
- Let \mathbf{Y} be any of the previous speaker identity variables sets.
- Let \mathbf{X}_d be the channel variables of the development set.
- Let \mathbf{X}_t be the channel variables of the test set.
- Let \mathbf{X} be any of the previous channel variables sets.
- Let $\mathbf{X}_i = [\mathbf{x}_{i1}, \mathbf{x}_{i2}, \dots, \mathbf{x}_{iH_i}]$ be the channel variables of speaker i .
- Let $\mathcal{M} = (\mu, \mathbf{V}, \mathbf{U}, \mathbf{D})$ be the set of all the model parameters.

2 Likelihood calculations

2.1 Definitions

We define the sufficient statistics for speaker i . The zero-order statistic is the number of observations of speaker i N_i . The first-order and second-order statistics are

$$\mathbf{F}_i = \sum_{j=1}^{H_i} \sum_{l=1}^{L_{ij}} \phi_{ijl} \quad (5)$$

$$\mathbf{S}_i = \sum_{j=1}^{H_i} \sum_{l=1}^{L_{ij}} \phi_{ijl} \phi_{ijl}^T \quad (6)$$

We define the centered statistics as

$$\bar{\mathbf{F}}_i = \mathbf{F}_i - N_i \mu \quad (7)$$

$$\bar{\mathbf{S}}_i = \sum_{j=1}^{H_i} \sum_{l=1}^{L_{ij}} (\phi_{ijl} - \mu) (\phi_{ijl} - \mu)^T = \mathbf{S}_i - \mu \mathbf{F}_i^T - \mathbf{F}_i \mu^T + N_i \mu \mu^T \quad (8)$$

We define the session statistics as

$$\mathbf{F}_{ij} = \sum_{l=1}^{L_{ij}} \phi_{ijl} \quad (9)$$

$$\bar{\mathbf{F}}_{ij} = \mathbf{F}_{ij} - L_{ij} \mu \quad (10)$$

where L_{ij} is the number of channels for the conversation ij .

We define the global statistics

$$N = \sum_{i=1}^M N_i \quad (11)$$

$$\mathbf{F} = \sum_{i=1}^M \mathbf{F}_i \quad (12)$$

$$\bar{\mathbf{F}} = \sum_{i=1}^M \bar{\mathbf{F}}_i \quad (13)$$

$$\mathbf{S} = \sum_{i=1}^M \mathbf{S}_i \quad (14)$$

$$\bar{\mathbf{S}} = \sum_{i=1}^M \bar{\mathbf{S}}_i \quad (15)$$

2.2 Data conditional likelihood

The likelihood of the data given the hidden variables for speaker i is

$$\ln P(\Phi_i | \mathbf{y}_i, \mathbf{X}_i, \mathcal{M}) = \sum_{j=1}^{H_i} \sum_{l=1}^{L_{ij}} \ln \mathcal{N}(\phi_{ijl} | \mu + \mathbf{V}\mathbf{y}_i + \mathbf{U}\mathbf{x}_{ij}, \mathbf{W}^{-1}) \quad (16)$$

$$= \frac{N_i}{2} \ln \left| \frac{\mathbf{W}}{2\pi} \right| - \frac{1}{2} \sum_{j=1}^{H_i} \sum_{l=1}^{L_{ij}} (\phi_{ijl} - \mu - \mathbf{V}\mathbf{y}_i - \mathbf{U}\mathbf{x}_{ij})^T \mathbf{W} (\phi_{ijl} - \mu - \mathbf{V}\mathbf{y}_i - \mathbf{U}\mathbf{x}_{ij}) \quad (17)$$

$$= \frac{N_i}{2} \ln \left| \frac{\mathbf{W}}{2\pi} \right| - \frac{1}{2} \text{tr}(\mathbf{W}\bar{\mathbf{S}}_i) + \mathbf{y}_i^T \mathbf{V}^T \mathbf{W} \bar{\mathbf{F}}_i - \frac{N_i}{2} \mathbf{y}_i^T \mathbf{V}^T \mathbf{W} \mathbf{V} \mathbf{y}_i \\ + \sum_{j=1}^{H_i} \mathbf{x}_{ij}^T \mathbf{U}^T \mathbf{W} \bar{\mathbf{F}}_{ij} - L_{ij} \mathbf{y}_i^T \mathbf{V}^T \mathbf{W} \mathbf{U} \mathbf{x}_{ij} - \frac{1}{2} L_{ij} \mathbf{x}_{ij}^T \mathbf{U}^T \mathbf{W} \mathbf{U} \mathbf{x}_{ij} \quad (18)$$

We can write this likelihood in other form if we define:

$$\tilde{\mathbf{y}}_{ij} = \begin{bmatrix} \mathbf{y}_i \\ \mathbf{x}_{ij} \\ 1 \end{bmatrix}, \quad \tilde{\mathbf{V}} = [\mathbf{V} \quad \mathbf{U} \quad \mu] \quad (19)$$

Then

$$\ln P(\Phi_i | \mathbf{y}_i, \mathbf{X}_i, \mathcal{M}) = \sum_{j=1}^{H_i} \sum_{l=1}^{L_{ij}} \ln \mathcal{N}(\phi_{ijl} | \tilde{\mathbf{V}} \tilde{\mathbf{y}}_{ij}, \mathbf{W}^{-1}) \quad (20)$$

$$= \frac{N_i}{2} \ln \left| \frac{\mathbf{W}}{2\pi} \right| - \frac{1}{2} \sum_{j=1}^{H_i} \sum_{l=1}^{L_{ij}} (\phi_{ijl} - \tilde{\mathbf{V}} \tilde{\mathbf{y}}_{ij})^T \mathbf{W} (\phi_{ijl} - \tilde{\mathbf{V}} \tilde{\mathbf{y}}_{ij}) \quad (21)$$

$$= \frac{N_i}{2} \ln \left| \frac{\mathbf{W}}{2\pi} \right| - \frac{1}{2} \text{tr}(\mathbf{W}\mathbf{S}_i) + \sum_{j=1}^{H_i} \tilde{\mathbf{y}}_{ij}^T \tilde{\mathbf{V}}^T \mathbf{W} \bar{\mathbf{F}}_{ij} - \frac{1}{2} L_{ij} \tilde{\mathbf{y}}_{ij}^T \tilde{\mathbf{V}}^T \mathbf{W} \tilde{\mathbf{V}} \tilde{\mathbf{y}}_{ij} \quad (22)$$

$$= \frac{N_i}{2} \ln \left| \frac{\mathbf{W}}{2\pi} \right| - \frac{1}{2} \text{tr} \left(\mathbf{W} \left(\mathbf{S}_i + \sum_{j=1}^{H_i} -2\mathbf{F}_{ij} \tilde{\mathbf{y}}_{ij}^T \tilde{\mathbf{V}}^T + L_{ij} \tilde{\mathbf{V}} \tilde{\mathbf{y}}_{ij} \tilde{\mathbf{y}}_{ij}^T \tilde{\mathbf{V}}^T \right) \right) \quad (23)$$

2.3 Posterior of the hidden variables

The posterior of the hidden variables can be decomposed into two factors:

$$P(\mathbf{y}_i, \mathbf{X}_i | \Phi_i, \mathcal{M}) = P(\mathbf{X}_i | \mathbf{y}_i, \Phi_i, \mathcal{M}) P(\mathbf{y}_i | \Phi_i, \mathcal{M}) \quad (24)$$

2.3.1 Conditional posterior of \mathbf{X}_i

The conditional posterior of \mathbf{X}_i is

$$P(\mathbf{X}_i | \mathbf{y}_i, \Phi_i, \mathcal{M}) = \frac{P(\Phi_i | \mathbf{y}_i, \mathbf{X}_i, \mathcal{M}) P(\mathbf{X}_i)}{P(\Phi_i | \mathbf{y}_i, \mathcal{M})} \quad (25)$$

Using equations (3) and (18)

$$\ln P(\mathbf{X}_i|\mathbf{y}_i, \Phi_i, \mathcal{M}) = \ln P(\Phi_i|\mathbf{y}_i, \mathbf{X}_i, \mathcal{M}) + \ln P(\mathbf{X}_i|\mathcal{M}) + \text{const} \quad (26)$$

$$= \sum_{j=1}^{H_i} \mathbf{x}_{ij}^T \mathbf{U}^T \mathbf{W} \bar{\mathbf{F}}_{ij} - L_{ij} \mathbf{y}^T \mathbf{V}^T \mathbf{W} \mathbf{U} \mathbf{x}_{ij} - \frac{1}{2} L_{ij} \mathbf{x}_{ij}^T \mathbf{U}^T \mathbf{W} \mathbf{U} \mathbf{x}_{ij} - \frac{1}{2} \mathbf{x}_{ij}^T \mathbf{x}_{ij} + \text{const} \quad (27)$$

$$= \sum_{j=1}^{H_i} \mathbf{x}_{ij}^T \mathbf{U}^T \mathbf{W} (\bar{\mathbf{F}}_{ij} - L_{ij} \mathbf{V} \mathbf{y}_i) - \frac{1}{2} \mathbf{x}_{ij}^T \mathbf{L}_{\mathbf{x}_{ij}} \mathbf{x}_{ij} + \text{const} \quad (28)$$

$$= \sum_{j=1}^{H_i} \mathbf{x}_{ij}^T \zeta_{ij} - \frac{1}{2} \mathbf{x}_{ij}^T \mathbf{L}_{\mathbf{x}_{ij}} \mathbf{x}_{ij} + \text{const} \quad (29)$$

where

$$\zeta_{ij} = \mathbf{U}^T \mathbf{W} (\bar{\mathbf{F}}_{ij} - L_{ij} \mathbf{V} \mathbf{y}_i) = \tilde{\zeta}_{ij} - L_{ij} \mathbf{J} \mathbf{y}_i \quad (30)$$

$$\tilde{\zeta}_{ij} = \mathbf{U}^T \mathbf{W} \bar{\mathbf{F}}_{ij} \quad (31)$$

$$\mathbf{J} = \mathbf{U}^T \mathbf{W} \mathbf{V} \quad (32)$$

$$\mathbf{L}_{\mathbf{x}_{ij}} = \mathbf{I} + L_{ij} \mathbf{U}^T \mathbf{W} \mathbf{U} \quad (33)$$

Equation (29) has the form of a product of Gaussian distributions. Therefore

$$P(\mathbf{X}_i|\mathbf{y}_i, \Phi_i, \mathcal{M}) = \prod_{j=1}^{H_i} \mathcal{N}(\mathbf{x}_{ij}|\bar{\mathbf{x}}_{ij}, \mathbf{L}_{\mathbf{x}_{ij}}^{-1}) \quad (34)$$

where

$$\bar{\mathbf{x}}_{ij} = \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \zeta_{ij} \quad (35)$$

2.3.2 Posterior of \mathbf{y}_i

The marginal posterior of \mathbf{y} is

$$P(\mathbf{y}|\Phi_i, \mathcal{M}) = \frac{P(\Phi_i|\mathbf{y}_i, \mathcal{M}) P(\mathbf{y})}{P(\Phi_i|\mathcal{M})} \quad (36)$$

We can use Bayes Theorem to write

$$P(\Phi_i, \mathbf{X}_i|\mathbf{y}_i, \mathcal{M}) = P(\Phi_i|\mathbf{y}_i, \mathbf{X}_i, \mathcal{M}) P(\mathbf{X}_i|\mathbf{y}_i, \mathcal{M}) = P(\mathbf{X}_i|\Phi_i, \mathbf{y}_i, \mathcal{M}) P(\Phi_i|\mathbf{y}_i, \mathcal{M}) \quad (37)$$

Simplifying

$$P(\Phi_i|\mathbf{y}_i, \mathbf{X}_i, \mathcal{M}) P(\mathbf{X}_i) = P(\mathbf{X}_i|\Phi_i, \mathbf{y}_i, \mathcal{M}) P(\Phi_i|\mathbf{y}_i, \mathcal{M}) \quad (38)$$

Then

$$P(\mathbf{y}|\Phi_i, \mathcal{M}) = \frac{P(\Phi_i|\mathbf{y}_i, \mathbf{X}_i, \mathcal{M}) P(\mathbf{X}_i) P(\mathbf{y})}{P(\mathbf{X}_i|\Phi_i, \mathbf{y}_i, \mathcal{M}) P(\Phi_i|\mathcal{M})} \Big|_{\mathbf{x}_i=0} \quad (39)$$

Using equations (2), (18) and (34)

$$\ln P(\mathbf{y}_i|\Phi_i, \mathcal{M}) = \ln P(\Phi_i|\mathbf{y}_i, \mathbf{X}_i, \mathcal{M}) + \ln P(\mathbf{y}) - \ln P(\mathbf{X}_i|\Phi_i, \mathbf{y}_i, \mathcal{M}) + \text{const} \quad (40)$$

$$= \mathbf{y}^T \mathbf{V}^T \mathbf{W} \bar{\mathbf{F}}_i - \frac{N_i}{2} \mathbf{y}_i^T \mathbf{V}^T \mathbf{W} \mathbf{V} \mathbf{y}_i - \frac{1}{2} \mathbf{y}_i^T \mathbf{y}_i + \frac{1}{2} \sum_{j=1}^{H_i} \bar{\mathbf{x}}_{ij}^T \mathbf{L}_{\mathbf{x}_{ij}} \bar{\mathbf{x}}_{ij} + \text{const} \quad (41)$$

$$= \mathbf{y}^T \mathbf{V}^T \mathbf{W} \bar{\mathbf{F}}_i - \frac{1}{2} \mathbf{y}_i^T (\mathbf{I} + N_i \mathbf{V}^T \mathbf{W} \mathbf{V}) \mathbf{y}_i + \frac{1}{2} \sum_{j=1}^{H_i} (\bar{\mathbf{F}}_{ij} - L_{ij} \mathbf{V} \mathbf{y}_i)^T \mathbf{W} \mathbf{U} \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \mathbf{U}^T \mathbf{W} (\bar{\mathbf{F}}_{ij} - L_{ij} \mathbf{V} \mathbf{y}_i) + \text{const} \quad (42)$$

$$= \mathbf{y}^T \mathbf{V}^T \mathbf{W} \bar{\mathbf{F}}_i - \frac{1}{2} \mathbf{y}_i^T (\mathbf{I} + N_i \mathbf{V}^T \mathbf{W} \mathbf{V}) \mathbf{y}_i + \frac{1}{2} \sum_{j=1}^{H_i} \bar{\mathbf{F}}_{ij}^T \mathbf{W} \mathbf{U} \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \mathbf{U}^T \mathbf{W} \bar{\mathbf{F}}_{ij} - 2L_{ij} \mathbf{y}_i^T \mathbf{V}^T \mathbf{W} \mathbf{U} \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \mathbf{U}^T \mathbf{W} \bar{\mathbf{F}}_{ij} + L_{ij}^2 \mathbf{y}_i^T \mathbf{V}^T \mathbf{W} \mathbf{U} \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \mathbf{U}^T \mathbf{W} \mathbf{V} \mathbf{y}_i + \text{const} \quad (43)$$

$$= \mathbf{y}^T \mathbf{V}^T \left(\mathbf{W} \bar{\mathbf{F}}_i - \sum_{j=1}^{H_i} L_{ij} \mathbf{W} \mathbf{U} \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \mathbf{U}^T \mathbf{W} \bar{\mathbf{F}}_{ij} \right) - \frac{1}{2} \mathbf{y}_i^T \left(\mathbf{I} + \mathbf{V}^T \left(N_i \mathbf{W} - \sum_{j=1}^{H_i} L_{ij}^2 \mathbf{W} \mathbf{U} \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \mathbf{U}^T \mathbf{W} \right) \mathbf{V} \right) \mathbf{y}_i + \text{const} \quad (44)$$

Then

$$P(\mathbf{y}_i|\Phi_i, \mathcal{M}) = \mathcal{N}(\mathbf{y}_i|\bar{\mathbf{y}}_i, \mathbf{L}_{\mathbf{y}_i}^{-1}) \quad (45)$$

where

$$\mathbf{L}_{\mathbf{y}_i} = \mathbf{I} + \mathbf{V}^T \left(N_i \mathbf{W} - \sum_{j=1}^{H_i} L_{ij}^2 \mathbf{W} \mathbf{U} \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \mathbf{U}^T \mathbf{W} \right) \mathbf{V} \quad (46)$$

$$= \mathbf{I} + N_i \mathbf{V}^T \mathbf{W} \mathbf{V} - \sum_{j=1}^{H_i} L_{ij}^2 \mathbf{J}^T \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \mathbf{J} \quad (47)$$

$$\gamma_i = \mathbf{V}^T \left(\mathbf{W} \bar{\mathbf{F}}_i - \sum_{j=1}^{H_i} L_{ij} \mathbf{W} \mathbf{U} \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \mathbf{U}^T \mathbf{W} \bar{\mathbf{F}}_{ij} \right) = \tilde{\gamma}_i - \sum_{j=1}^{H_i} L_{ij} \mathbf{J}^T \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \tilde{\zeta}_{ij} \quad (48)$$

$$\tilde{\gamma}_i = \mathbf{V}^T \mathbf{W} \bar{\mathbf{F}}_i \quad (49)$$

$$\bar{\mathbf{y}}_i = \mathbf{L}_{\mathbf{y}_i}^{-1} \gamma_i \quad (50)$$

2.4 Marginal likelihood of the data

The marginal likelihood of the data is

$$P(\Phi_i|\mathcal{M}) = \frac{P(\Phi_i|\mathbf{y}_i, \mathbf{X}_i, \mathcal{M}) P(\mathbf{y}_i) P(\mathbf{X}_i)}{P(\mathbf{X}_i|\mathbf{y}_i, \Phi_i, \mathcal{M}) P(\mathbf{y}_i|\Phi_i, \mathcal{M})} \Big|_{\mathbf{y}_i=0, \mathbf{X}_i=0} \quad (51)$$

Taking equations (18), (2), (3) (34) and (45)

$$\ln P(\Phi_i|\mathcal{M}) = \frac{N_i}{2} \ln \left| \frac{\mathbf{W}}{2\pi} \right| - \frac{1}{2} \text{tr}(\mathbf{W} \bar{\mathbf{S}}_i) - \frac{1}{2} \sum_{j=1}^{H_i} \ln |\mathbf{L}_{\mathbf{x}_{ij}}| + \frac{1}{2} \sum_{j=1}^{H_i} \tilde{\zeta}_{ij}^T \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \tilde{\zeta}_{ij} - \frac{1}{2} \ln |\mathbf{L}_{\mathbf{y}_i}| + \frac{1}{2} \gamma_i^T \mathbf{L}_{\mathbf{y}_i}^{-1} \gamma_i \quad (52)$$

3 EM algorithm

3.1 E-step

In the E-step we calculate the posterior of \mathbf{y} and \mathbf{X} with equation (24)

3.2 M-step ML

We maximize the EM auxiliary function $\mathcal{Q}(\mathcal{M})$

$$\mathcal{Q}(\mathcal{M}) = \sum_{i=1}^M \mathbb{E}_{\mathbf{Y}, \mathbf{X}} [\ln P(\Phi_i, \mathbf{y}_i, \mathbf{X}_i | \mathcal{M})] \quad (53)$$

$$= \sum_{i=1}^M \mathbb{E}_{\mathbf{Y}, \mathbf{X}} [\ln P(\Phi_i | \mathbf{y}_i, \mathbf{X}_i, \mathcal{M})] + \mathbb{E}_{\mathbf{Y}, \mathbf{X}} [\ln P(\mathbf{y}_i)] + \mathbb{E}_{\mathbf{Y}, \mathbf{X}} [\ln P(\mathbf{X}_i)] \quad (54)$$

Taking equation (23)

$$\mathcal{Q}(\mathcal{M}) = \frac{N}{2} \ln \left| \frac{\mathbf{W}}{2\pi} \right| - \frac{1}{2} \text{tr} \left(\mathbf{W} \left(\mathbf{S} + \sum_{i=1}^M \sum_{j=1}^{H_i} -2\mathbf{F}_{ij} \mathbb{E}_{\mathbf{Y}, \mathbf{X}} [\tilde{\mathbf{y}}_{ij}]^T \tilde{\mathbf{V}}^T + L_{ij} \tilde{\mathbf{V}} \mathbb{E}_{\mathbf{Y}, \mathbf{X}} [\tilde{\mathbf{y}}_{ij} \tilde{\mathbf{y}}_{ij}^T] \tilde{\mathbf{V}}^T \right) \right) \quad (55)$$

we define

$$\mathbf{R}_{\tilde{\mathbf{y}}} = \sum_{i=1}^M \sum_{j=1}^{H_i} L_{ij} \mathbb{E}_{\mathbf{Y}} [\tilde{\mathbf{y}}_{ij} \tilde{\mathbf{y}}_{ij}^T] \quad (56)$$

$$\mathbf{C} = \sum_{i=1}^M \sum_{j=1}^{H_i} \mathbf{F}_{ij} \mathbb{E}_{\mathbf{Y}} [\tilde{\mathbf{y}}_{ij}]^T \quad (57)$$

then

$$\mathcal{Q}(\mathcal{M}) = \frac{N}{2} \ln |\mathbf{W}| - \frac{1}{2} \text{tr} \left(\mathbf{W} \left(\mathbf{S} - 2\mathbf{C} \tilde{\mathbf{V}}^T + \tilde{\mathbf{V}} \mathbf{R}_{\tilde{\mathbf{y}}} \tilde{\mathbf{V}}^T \right) \right) + \text{const} \quad (58)$$

$$\frac{\partial \mathcal{Q}(\mathcal{M})}{\partial \tilde{\mathbf{V}}} = \mathbf{C} - \tilde{\mathbf{V}} \mathbf{R}_{\tilde{\mathbf{y}}} = \mathbf{0} \implies \quad (59)$$

$$\tilde{\mathbf{V}} = \mathbf{C} \mathbf{R}_{\tilde{\mathbf{y}}}^{-1} \quad (60)$$

$$\frac{\partial \mathcal{Q}(\mathcal{M})}{\partial \mathbf{W}} = \frac{N}{2} (2\mathbf{W}^{-1} - \text{diag}(\mathbf{W}^{-1})) - \frac{1}{2} (\mathbf{K} + \mathbf{K}^T - \text{diag}(\mathbf{K})) = \mathbf{0} \quad (61)$$

where $\mathbf{K} = \mathbf{S} - 2\mathbf{C} \tilde{\mathbf{V}}^T + \tilde{\mathbf{V}} \mathbf{R}_{\tilde{\mathbf{y}}} \tilde{\mathbf{V}}^T$, so

$$\mathbf{W}^{-1} = \frac{1}{N} \frac{\mathbf{K} + \mathbf{K}^T}{2} \quad (62)$$

$$= \frac{1}{N} (\mathbf{S}_{\phi} - \tilde{\mathbf{V}} \mathbf{C}^T - \mathbf{C} \tilde{\mathbf{V}}^T + \tilde{\mathbf{V}} \mathbf{R}_{\tilde{\mathbf{y}}} \tilde{\mathbf{V}}^T) \quad (63)$$

$$= \frac{1}{N} (\mathbf{S} - \tilde{\mathbf{V}} \mathbf{C}^T) \quad (64)$$

Finally, we need to evaluate the expectations $\mathbb{E}_{\mathbf{Y}} [\tilde{\mathbf{y}}_{ij}]$ and $\mathbb{E}_{\mathbf{Y}} [\tilde{\mathbf{y}}_{ij} \tilde{\mathbf{y}}_{ij}^T]$ and compute $\mathbf{R}_{\tilde{\mathbf{y}}}$ and \mathbf{C} .

$$\mathbf{C} = \sum_{i=1}^M \sum_{j=1}^{H_i} \mathbf{F}_{ij} \mathbb{E}_{\mathbf{Y}, \mathbf{X}} [\tilde{\mathbf{y}}_{ij}]^T = \sum_{i=1}^M \sum_{j=1}^{H_i} \mathbf{F}_{ij} \begin{bmatrix} \mathbb{E}_{\mathbf{Y}} [\mathbf{y}_i] \\ \mathbb{E}_{\mathbf{Y}, \mathbf{X}} [\mathbf{x}_{ij}] \\ 1 \end{bmatrix}^T = [\mathbf{C}_{\mathbf{y}} \quad \mathbf{C}_{\mathbf{x}} \quad \mathbf{F}] \quad (65)$$

Now

$$\mathbf{E}_{\mathbf{Y}} [\mathbf{y}_i] = \bar{\mathbf{y}}_i \quad (66)$$

$$\mathbf{E}_{\mathbf{Y}, \mathbf{X}} [\mathbf{x}_{ij}] = \mathbf{E}_{\mathbf{Y}} [\bar{\mathbf{x}}_{ij}] = \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \left(\tilde{\zeta}_{ij} - L_{ij} \mathbf{J} \bar{\mathbf{y}}_i \right) \quad (67)$$

$$\mathbf{C}_{\mathbf{y}} = \sum_{i=1}^M \sum_{j=1}^{H_i} \mathbf{F}_{ij} \bar{\mathbf{y}}_i^T = \sum_{i=1}^M \mathbf{F}_i \bar{\mathbf{y}}_i^T \quad (68)$$

$$\mathbf{C}_{\mathbf{x}} = \sum_{i=1}^M \sum_{j=1}^{H_i} \mathbf{F}_{ij} \left(\tilde{\zeta}_{ij} - L_{ij} \mathbf{J} \bar{\mathbf{y}}_i \right)^T \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \quad (69)$$

$$\mathbf{R}_{\tilde{\mathbf{y}}} = \begin{bmatrix} \mathbf{R}_{\mathbf{y}} & \mathbf{R}_{\mathbf{yx}} & \mathbf{R}_{\mathbf{y1}} \\ \mathbf{R}_{\mathbf{xy}} & \mathbf{R}_{\mathbf{x}} & \mathbf{R}_{\mathbf{x1}} \\ \mathbf{R}_{\mathbf{y1}}^T & \mathbf{R}_{\mathbf{x1}}^T & N \end{bmatrix} \quad (70)$$

Now

$$\mathbf{R}_{\mathbf{y1}} = \sum_{i=1}^M N_i \mathbf{E}_{\mathbf{Y}} [\mathbf{y}_i] = \sum_{i=1}^M N_i \bar{\mathbf{y}}_i \quad (71)$$

$$\mathbf{R}_{\mathbf{x1}} = \sum_{i=1}^M \sum_{j=1}^{H_i} L_{ij} \mathbf{E}_{\mathbf{Y}, \mathbf{X}} [\mathbf{x}_{ij}] = \sum_{i=1}^M \sum_{j=1}^{H_i} L_{ij} \mathbf{L}_{\mathbf{x}_{ij}}^{-1} (\mathbf{U}^T \mathbf{W} \bar{\mathbf{F}}_{ij} - L_{ij} \mathbf{J} \bar{\mathbf{y}}_i) \quad (72)$$

$$\mathbf{R}_{\mathbf{y}} = \sum_{i=1}^M N_i \mathbf{E}_{\mathbf{Y}} [\mathbf{y}_i \mathbf{y}_i^T] = \sum_{i=1}^M N_i (\mathbf{L}_{\mathbf{y}_i}^{-1} + \bar{\mathbf{y}}_i \bar{\mathbf{y}}_i^T) \quad (73)$$

$$\mathbf{R}_{\mathbf{xy}} = \sum_{i=1}^M \sum_{j=1}^{H_i} L_{ij} \mathbf{E}_{\mathbf{Y}, \mathbf{X}} [\mathbf{x}_{ij} \mathbf{y}_i^T] = \sum_{i=1}^M \sum_{j=1}^{H_i} L_{ij} \mathbf{E}_{\mathbf{Y}} \left[\mathbf{L}_{\mathbf{x}_{ij}}^{-1} \left(\tilde{\zeta}_{ij} - L_{ij} \mathbf{J} \mathbf{y}_i \right) \mathbf{y}_i^T \right] \quad (74)$$

$$= \sum_{i=1}^M \sum_{j=1}^{H_i} L_{ij} \mathbf{L}_{\mathbf{x}_{ij}}^{-1} (\mathbf{U}^T \mathbf{W} \bar{\mathbf{F}}_{ij} \bar{\mathbf{y}}_i^T - L_{ij} \mathbf{J} \mathbf{E}_{\mathbf{Y}} [\mathbf{y}_i \mathbf{y}_i^T]) \quad (75)$$

$$\mathbf{R}_{\mathbf{x}} = \sum_{i=1}^M \sum_{j=1}^{H_i} L_{ij} \mathbf{E}_{\mathbf{Y}, \mathbf{X}} [\mathbf{x}_{ij} \mathbf{x}_{ij}^T] \quad (76)$$

$$= \sum_{i=1}^M \sum_{j=1}^{H_i} L_{ij} \left(\mathbf{L}_{\mathbf{x}_{ij}}^{-1} + \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \mathbf{E}_{\mathbf{Y}} \left[(\mathbf{U}^T \mathbf{W} \bar{\mathbf{F}}_{ij} - L_{ij} \mathbf{J} \mathbf{y}_i) (\mathbf{U}^T \mathbf{W} \bar{\mathbf{F}}_{ij} - L_{ij} \mathbf{J} \mathbf{y}_i)^T \right] \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \right) \quad (77)$$

$$\begin{aligned} &= \sum_{i=1}^M \sum_{j=1}^{H_i} L_{ij} \left(\mathbf{L}_{\mathbf{x}_{ij}}^{-1} + \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \left(\mathbf{U}^T \mathbf{W} \bar{\mathbf{F}}_{ij} \bar{\mathbf{F}}_{ij}^T \mathbf{W} \mathbf{U} \right. \right. \\ &\quad \left. \left. - L_{ij} \mathbf{U}^T \mathbf{W} \bar{\mathbf{F}}_{ij} \bar{\mathbf{y}}_i^T \mathbf{J}^T - L_{ij} \mathbf{J} \bar{\mathbf{y}}_i \bar{\mathbf{F}}_{ij}^T \mathbf{W} \mathbf{U} \right. \right. \\ &\quad \left. \left. + L_{ij}^2 \mathbf{J} \mathbf{E}_{\mathbf{Y}} [\mathbf{y}_i \mathbf{y}_i^T] \mathbf{J}^T \right) \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \right) \end{aligned} \quad (78)$$

3.3 M-step MD

We assume a more general prior for the hidden variables:

$$P(\mathbf{y}_i) = \mathcal{N}(\mathbf{y}_i | \mu_{\mathbf{y}}, \mathbf{\Lambda}_{\mathbf{y}}^{-1}) \quad (79)$$

$$P(\mathbf{x}_{ij} | \mathbf{y}_i) = \mathcal{N}(\mathbf{x}_{ij} | \mathbf{H} \mathbf{y}_i + \mu_{\mathbf{x}}, \mathbf{\Lambda}_{\mathbf{x}}^{-1}) \quad (80)$$

To minimize the divergence we maximize

$$\mathcal{Q}(\mu_{\mathbf{y}}, \mathbf{\Lambda}_{\mathbf{y}}, \mathbf{H}, \mu_{\mathbf{x}}, \mathbf{\Lambda}_{\mathbf{x}}) = \sum_{i=1}^M \mathbb{E}_{\mathbf{Y}} [\ln \mathcal{N}(\mathbf{y}_i | \mu_{\mathbf{y}}, \mathbf{\Lambda}_{\mathbf{y}}^{-1})] + \sum_{j=1}^{H_i} \mathbb{E}_{\mathbf{Y}, \mathbf{X}} [\ln \mathcal{N}(\mathbf{x}_{ij} | \mathbf{H}\mathbf{y}_i + \mu_{\mathbf{x}}, \mathbf{\Lambda}_{\mathbf{x}}^{-1})] \quad (81)$$

$$\begin{aligned} &= \frac{M}{2} \ln |\mathbf{\Lambda}_{\mathbf{y}}| - \frac{1}{2} \text{tr} \left(\mathbf{\Lambda}_{\mathbf{y}} \sum_{i=1}^M \mathbb{E}_{\mathbf{Y}} [(\mathbf{y}_i - \mu_{\mathbf{y}})(\mathbf{y}_i - \mu_{\mathbf{y}})^T] \right) \\ &\quad + \frac{H}{2} \ln |\mathbf{\Lambda}_{\mathbf{x}}| - \frac{1}{2} \text{tr} \left(\mathbf{\Lambda}_{\mathbf{x}} \sum_{i=1}^M \sum_{j=1}^{H_i} \mathbb{E}_{\mathbf{Y}, \mathbf{X}} [(\mathbf{x}_{ij} - \mathbf{H}\mathbf{y}_i - \mu_{\mathbf{x}})(\mathbf{x}_{ij} - \mathbf{H}\mathbf{y}_i - \mu_{\mathbf{x}})^T] \right) \\ &\quad + \text{const} \end{aligned} \quad (82)$$

$$\frac{\partial \mathcal{Q}(\mu_{\mathbf{y}}, \mathbf{\Lambda}_{\mathbf{y}}, \mathbf{H}, \mu_{\mathbf{x}}, \mathbf{\Lambda}_{\mathbf{x}})}{\partial \mu_{\mathbf{y}}} = \frac{1}{2} \sum_{i=1}^M \mathbf{\Lambda}_{\mathbf{y}} \mathbb{E}_{\mathbf{Y}} [\mathbf{y}_i - \mu_{\mathbf{y}}] = \mathbf{0} \implies \quad (83)$$

$$\mu_{\mathbf{y}} = \frac{1}{M} \sum_{i=1}^M \mathbb{E}_{\mathbf{Y}} [\mathbf{y}_i] \quad (84)$$

$$\frac{\partial \mathcal{Q}(\mu_{\mathbf{y}}, \mathbf{\Lambda}_{\mathbf{y}}, \mathbf{H}, \mu_{\mathbf{x}}, \mathbf{\Lambda}_{\mathbf{x}})}{\partial \mathbf{\Lambda}_{\mathbf{y}}} = \frac{M}{2} (2\mathbf{\Lambda}_{\mathbf{y}}^{-1} - \text{diag}(\mathbf{\Lambda}_{\mathbf{y}}^{-1})) - \frac{1}{2} (2\mathbf{S} - \text{diag}(\mathbf{S})) = \mathbf{0} \quad (85)$$

where $\mathbf{S} = \sum_{i=1}^M \mathbb{E}_{\mathbf{Y}} [(\mathbf{y}_i - \mu_{\mathbf{y}})(\mathbf{y}_i - \mu_{\mathbf{y}})^T]$, so

$$\mathbf{\Sigma}_{\mathbf{y}} = \mathbf{\Lambda}_{\mathbf{y}}^{-1} \quad (86)$$

$$= \frac{1}{M} \sum_{i=1}^M \mathbb{E}_{\mathbf{Y}} [(\mathbf{y}_i - \mu_{\mathbf{y}})(\mathbf{y}_i - \mu_{\mathbf{y}})^T] \quad (87)$$

$$= \frac{1}{M} \sum_{i=1}^M \mathbb{E}_{\mathbf{Y}} [\mathbf{y}_i \mathbf{y}_i^T] - \mu_{\mathbf{y}} \mathbb{E}_{\mathbf{Y}} [\mathbf{y}_i]^T - \mathbb{E}_{\mathbf{Y}} [\mathbf{y}_i] \mu_{\mathbf{y}}^T + \mu_{\mathbf{y}} \mu_{\mathbf{y}}^T \quad (88)$$

$$= \frac{1}{M} \sum_{i=1}^M \mathbb{E}_{\mathbf{Y}} [\mathbf{y}_i \mathbf{y}_i^T] - \mu_{\mathbf{y}} \mu_{\mathbf{y}}^T \quad (89)$$

$$\frac{\partial \mathcal{Q}(\mu_{\mathbf{y}}, \mathbf{\Lambda}_{\mathbf{y}}, \mathbf{H}, \mu_{\mathbf{x}}, \mathbf{\Lambda}_{\mathbf{x}})}{\partial \mu_{\mathbf{x}}} = \mathbf{\Lambda}_{\mathbf{x}} \sum_{i=1}^M \sum_{j=1}^{H_i} \mathbb{E}_{\mathbf{Y}, \mathbf{X}} [\mathbf{x}_{ij} - \mathbf{H}\mathbf{y}_i - \mu_{\mathbf{x}}] = \mathbf{0} \implies \quad (90)$$

$$\mu_{\mathbf{x}} = \frac{1}{H} \left(\sum_{i=1}^M \sum_{j=1}^{H_i} \mathbb{E}_{\mathbf{X}} [\mathbf{x}_{ij}] - \mathbf{H} \sum_{i=1}^M H_i \mathbb{E}_{\mathbf{Y}} [\mathbf{y}_i] \right) \quad (91)$$

$$= \frac{1}{H} (\mathbf{P}_{\mathbf{x}\mathbf{1}} - \mathbf{H} \mathbf{P}_{\mathbf{y}\mathbf{1}}) \quad (92)$$

where

$$\mathbf{P}_{\mathbf{x}\mathbf{1}} = \sum_{i=1}^M \sum_{j=1}^{H_i} \mathbb{E}_{\mathbf{X}} [\mathbf{x}_{ij}] \quad (93)$$

$$\mathbf{P}_{\mathbf{y}\mathbf{1}} = \sum_{i=1}^M H_i \mathbb{E}_{\mathbf{Y}} [\mathbf{y}_i] \quad (94)$$

$$\frac{\partial \mathcal{Q}(\mu_{\mathbf{y}}, \Lambda_{\mathbf{y}}, \mathbf{H}, \mu_{\mathbf{x}}, \Lambda_{\mathbf{x}})}{\partial \mathbf{H}} = \Lambda_{\mathbf{x}} \sum_{i=1}^M \sum_{j=1}^{H_i} \mathbf{E}_{\mathbf{Y}, \mathbf{X}} [(\mathbf{x}_{ij} - \mathbf{H}\mathbf{y}_i - \mu_{\mathbf{x}}) \mathbf{y}_i^T] = \mathbf{0} \quad (95)$$

$$\implies \mathbf{P}_{\mathbf{xy}} - \mathbf{H}\mathbf{P}_{\mathbf{y}} - \mu_{\mathbf{x}}\mathbf{P}_{1\mathbf{y}} = \mathbf{0} \quad (96)$$

$$\implies \mathbf{P}_{\mathbf{xy}} - \mathbf{H}\mathbf{P}_{\mathbf{y}} - \frac{1}{H}(\mathbf{P}_{\mathbf{x}1} - \mathbf{H}\mathbf{P}_{\mathbf{y}1})\mathbf{P}_{1\mathbf{y}} = \mathbf{0} \quad (97)$$

$$\implies \mathbf{P}_{\mathbf{xy}} - \frac{1}{H}\mathbf{P}_{\mathbf{x}1}\mathbf{P}_{1\mathbf{y}} - \mathbf{H}\left(\mathbf{P}_{\mathbf{y}} - \frac{1}{H}\mathbf{P}_{\mathbf{y}1}\mathbf{P}_{1\mathbf{y}}\right) = \mathbf{0} \implies \quad (98)$$

$$\mathbf{H} = \left(\mathbf{P}_{\mathbf{xy}} - \frac{1}{H}\mathbf{P}_{\mathbf{x}1}\mathbf{P}_{1\mathbf{y}}\right) \left(\mathbf{P}_{\mathbf{y}} - \frac{1}{H}\mathbf{P}_{\mathbf{y}1}\mathbf{P}_{1\mathbf{y}}\right)^{-1} \quad (99)$$

where

$$\mathbf{P}_{\mathbf{xy}} = \sum_{i=1}^M \sum_{j=1}^{H_i} \mathbf{E}_{\mathbf{Y}, \mathbf{X}} [\mathbf{x}_{ij}\mathbf{y}_i^T] \quad (100)$$

$$\mathbf{P}_{\mathbf{y}} = \sum_{i=1}^M H_i \mathbf{E}_{\mathbf{Y}} [\mathbf{y}_i\mathbf{y}_i^T] \quad (101)$$

$$\mathbf{P}_{\mathbf{x}} = \sum_{i=1}^M \sum_{j=1}^{H_i} \mathbf{E}_{\mathbf{X}} [\mathbf{x}_{ij}\mathbf{x}_{ij}^T] \quad (102)$$

$$\frac{\partial \mathcal{Q}(\mu_{\mathbf{y}}, \Lambda_{\mathbf{y}}, \mathbf{H}, \mu_{\mathbf{x}}, \Lambda_{\mathbf{x}})}{\partial \Lambda_{\mathbf{x}}} = \frac{H}{2} (2\Lambda_{\mathbf{x}}^{-1} - \text{diag}(\Lambda_{\mathbf{x}}^{-1})) - \frac{1}{2} (2\mathbf{S} - \text{diag}(\mathbf{S})) = \mathbf{0} \quad (103)$$

where $\mathbf{S} = \sum_{i=1}^M \sum_{j=1}^{H_i} \mathbf{E}_{\mathbf{Y}, \mathbf{X}} [(\mathbf{x}_{ij} - \mathbf{H}\mathbf{y}_i - \mu_{\mathbf{x}})(\mathbf{x}_{ij} - \mathbf{H}\mathbf{y}_i - \mu_{\mathbf{x}})^T]$, so

$$\Sigma_{\mathbf{x}} = \Lambda_{\mathbf{x}}^{-1} \quad (104)$$

$$\begin{aligned} &= \frac{1}{H} (\mathbf{P}_{\mathbf{x}} - \mathbf{P}_{\mathbf{xy}}\mathbf{H}^T - \mathbf{H}\mathbf{P}_{\mathbf{xy}}^T - \mathbf{P}_{\mathbf{x}1}\mu_{\mathbf{x}}^T - \mu_{\mathbf{x}}\mathbf{P}_{\mathbf{x}1}^T + \mathbf{H}\mathbf{P}_{\mathbf{y}}\mathbf{H}^T \\ &\quad + \mathbf{H}\mathbf{P}_{\mathbf{y}1}\mu_{\mathbf{x}}^T + \mu_{\mathbf{x}}\mathbf{P}_{\mathbf{y}1}^T\mathbf{H}^T + H\mu_{\mathbf{x}}\mu_{\mathbf{x}}^T) \end{aligned} \quad (105)$$

$$\begin{aligned} &= \frac{1}{H} (\mathbf{P}_{\mathbf{x}} - \mathbf{P}_{\mathbf{xy}}\mathbf{H}^T - \mathbf{H}\mathbf{P}_{\mathbf{xy}}^T + \mathbf{H}\mathbf{P}_{\mathbf{y}}\mathbf{H}^T \\ &\quad - (\mathbf{P}_{\mathbf{x}1} - \mathbf{H}\mathbf{P}_{\mathbf{y}1})\mu_{\mathbf{x}}^T - \mu_{\mathbf{x}}(\mathbf{P}_{\mathbf{x}1} - \mathbf{H}\mathbf{P}_{\mathbf{y}1})^T + H\mu_{\mathbf{x}}\mu_{\mathbf{x}}^T) \end{aligned} \quad (106)$$

$$= \frac{1}{H} (\mathbf{P}_{\mathbf{x}} - \mathbf{P}_{\mathbf{xy}}\mathbf{H}^T - \mathbf{H}\mathbf{P}_{\mathbf{xy}}^T + \mathbf{H}\mathbf{P}_{\mathbf{y}}\mathbf{H}^T - (\mathbf{P}_{\mathbf{x}1} - \mathbf{H}\mathbf{P}_{\mathbf{y}1})\mu_{\mathbf{x}}^T) \quad (107)$$

The transform $(\mathbf{y}, \mathbf{x}) = \phi(\mathbf{y}', \mathbf{x}')$ such as \mathbf{y}' and \mathbf{x}' has a standard prior is

$$\mathbf{y} = \mu_{\mathbf{y}} + (\Sigma_{\mathbf{y}}^{1/2})^T \mathbf{y}' \quad (108)$$

$$\mathbf{x} = \mu_{\mathbf{x}} + \mathbf{H}\mathbf{y} + (\Sigma_{\mathbf{x}}^{1/2})^T \mathbf{x}' \quad (109)$$

$$= \mu_{\mathbf{x}} + \mathbf{H}\mu_{\mathbf{y}} + \mathbf{H}(\Sigma_{\mathbf{y}}^{1/2})^T \mathbf{y}' + (\Sigma_{\mathbf{x}}^{1/2})^T \mathbf{x}' \quad (110)$$

We can transform μ , \mathbf{V} and \mathbf{U} using that transform

$$\mathbf{U}' = \mathbf{U}(\Sigma_{\mathbf{x}}^{-1/2})^T \quad (111)$$

$$\mathbf{V}' = (\mathbf{V} + \mathbf{U}\mathbf{H})(\Sigma_{\mathbf{y}}^{-1/2})^T \quad (112)$$

$$\mu' = \mu + (\mathbf{V} + \mathbf{U}\mathbf{H})\mu_{\mathbf{y}} + \mathbf{U}\mu_{\mathbf{x}} \quad (113)$$

3.4 Objective function

The EM objective function is equation (52) summed for all speakers

$$\begin{aligned} \ln P(\Phi|\mathcal{M}) = & \frac{N}{2} \ln \left| \frac{\mathbf{W}}{2\pi} \right| - \frac{1}{2} \text{tr}(\mathbf{W}\bar{\mathbf{S}}) - \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^{H_i} \ln |\mathbf{L}_{\mathbf{x}_{ij}}| + \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^{H_i} \tilde{\zeta}_{ij}^T \mathbf{L}_{\mathbf{x}_{ij}}^{-1} \tilde{\zeta}_{ij} \\ & - \frac{1}{2} \sum_{i=1}^M \ln |\mathbf{L}_{\mathbf{y}_i}| + \frac{1}{2} \sum_{i=1}^M \gamma_i^T \mathbf{L}_{\mathbf{y}_i}^{-1} \gamma_i \end{aligned} \quad (114)$$

4 Likelihood ratio

Given a model \mathcal{M} we can calculate the ratio of the posterior probabilities of target and non target as shown in [1]:

$$\frac{P(\mathcal{T}|\Phi_t, \mathcal{M}, \pi)}{P(\mathcal{N}|\Phi_t, \mathcal{M}, \pi)} = \frac{P_{\mathcal{T}}}{P_{\mathcal{N}}} \frac{P(\Phi_t|\mathcal{T}, \mathcal{M})}{P(\Phi_t|\mathcal{N}, \mathcal{M})} = \frac{P_{\mathcal{T}}}{P_{\mathcal{N}}} R(\Phi_t, \mathcal{M}) \quad (115)$$

where we have defined the plug-in likelihood ratio $R(\Phi_t, \mathcal{M})$. To get this ratio we need to calculate $P(\Phi|\theta, \mathcal{M})$. Given a model \mathcal{M} , the $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_M \in \mathbf{Y}$ are sampled independently from $P(\mathbf{y}|\mathcal{M})$. Besides, given the \mathcal{M} and a speaker i the set Φ_i of i-vectors produced by that speaker are drawn independently from $P(\Phi|\mathbf{y}_i, \mathcal{M})$. Using these independence assumptions we can write:

$$P(\Phi|\theta, \mathcal{M}) = \prod_{i=1}^M P(\Phi_i|\mathcal{M}) \quad (116)$$

$$P(\Phi_i|\mathbf{y}, \mathcal{M}) = \prod_{\phi \in \Phi_i} P(\phi|\mathbf{y}, \mathcal{M}) \quad (117)$$

Then, the likelihood of Φ is

$$P(\Phi|\theta, \mathcal{M}) = \prod_{i=1}^M \frac{P(\Phi_i|\mathbf{y}_0, \mathcal{M}) P(\mathbf{y}_0|\mathcal{M})}{P(\mathbf{y}_0|\Phi_i, \mathcal{M})} = K(\Phi) L(\theta|\Phi) \quad (118)$$

where $K(\Phi) = \prod_{i=1}^M P(\phi_j|\mathbf{y}_0, \mathcal{M})$ is a term that only dependent on the dataset, not θ , so it vanishes when doing the ratio and we do not need to calculate it. What we need to calculate is:

$$L(\theta|\Phi) = \prod_{i=1}^M Q(\Phi_i) \quad (119)$$

$$Q(\Phi_i) = \frac{P(\mathbf{y}_0|\mathcal{M})}{P(\mathbf{y}_0|\Phi_i, \mathcal{M})} \quad (120)$$

and the likelihood ratio is:

$$R(\Phi_t, \mathcal{M}) = \frac{Q(\{l, r\})}{Q(\{l\}) Q(\{r\})} \quad (121)$$

Making $\mathbf{y}_0 = 0$ we can get use (39), (2) to calculate $Q(\Phi)$

$$\ln Q(\Phi_i) = \frac{1}{2} (-\ln |\mathbf{L}_{\mathbf{y}_i}| + \gamma_i^T \mathbf{L}_{\mathbf{y}_i}^{-1} \gamma_i) \quad (122)$$

Given a set of training observations Φ_1 of a speaker 1 with statistics N_1 and $\bar{\mathbf{F}}_1$; and a set of test observations Φ_2 of a speaker 2 with statistics N_2 and $\bar{\mathbf{F}}_2$. To test if the speakers 1 and 2 are the same speaker the log-likelihood ratio is

$$\ln R(\Phi_t, \mathcal{M}) = \frac{1}{2} (-\ln |\mathbf{L}_3| + \gamma_3^T \mathbf{L}_3^{-1} \gamma_3 + \ln |\mathbf{L}_1| - \gamma_1^T \mathbf{L}_1^{-1} \gamma_1 + \ln |\mathbf{L}_2| - \gamma_2^T \mathbf{L}_2^{-1} \gamma_2) \quad (123)$$

where

$$P(\mathbf{y}|\Phi_1, \mathcal{M}) = \mathcal{N}(\mathbf{y}|\gamma_1 \mathbf{L}_1^{-1}, \mathbf{L}_1^{-1}) \quad (124)$$

$$P(\mathbf{y}|\Phi_2, \mathcal{M}) = \mathcal{N}(\mathbf{y}|\gamma_2 \mathbf{L}_2^{-1}, \mathbf{L}_2^{-1}) \quad (125)$$

$$P(\mathbf{y}|\Phi_1, \Phi_2, \mathcal{M}) = \mathcal{N}(\mathbf{y}|\gamma_3 \mathbf{L}_3^{-1}, \mathbf{L}_3^{-1}) \quad (126)$$

$$(127)$$

Using that $\gamma_3 = \gamma_1 + \gamma_2$:

$$\ln R(\Phi_t, \mathcal{M}) = \frac{1}{2} (\ln |\mathbf{L}_1| + \ln |\mathbf{L}_2| - \ln |\mathbf{L}_3| + 2\gamma_1^T \mathbf{L}_3^{-1} \gamma_2 + \gamma_1^T (\mathbf{L}_3^{-1} - \mathbf{L}_1^{-1}) \gamma_1 + \gamma_2^T (\mathbf{L}_3^{-1} - \mathbf{L}_2^{-1}) \gamma_2) \quad (128)$$

References

- [1] Niko Brummer and Edward De Villiers, “The Speaker Partitioning Problem,” in *Odyssey Speaker and Language Recognition Workshop*, Brno, Czech Republic, 2010.